Relevance Logic: Problems Open and Closed
Vienna Summer of Logic

Alasdair Urquhart

University of Toronto

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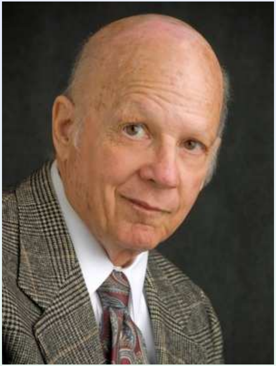
Alan Ross Anderson



Nuel Belnap

The relevance logic tradition, starting with Ackermann and continuing with the work of Anderson, Belnap and their students, has nurtured a fascinating collection of open problems that have stimulated research in the area for decades. Alan Ross Anderson provided an early collection of open problems in a paper from 1963.

The first in his list of problems is that of showing that the rule γ (from A and $\neg A \vee B$ to infer B) is admissible in **E**. The second problem is that of finding a decision procedure for **E**. The last problem is that of finding appropriate semantics for the whole of **E**, with an appropriate completeness theorem. All of these problems were subsequently solved.

The aim of the present talk is to present a set of unsolved problems in the area, as well as to report the very recent solution of a long-standing open problem in the area. If the health of an area of logic is to be judged by continued activity surrounding open problems, then relevance logic must be accounted a healthy area!

Decision problem for the semilattice system

I published a semantics based on semilattice models in 1972. The language for this semantics is $\to, \wedge, \vee$, that is to say, the language of the positive relevance logic $\mathbf{R}$. Let $\langle S, \cup, 0 \rangle$ be a semilattice with zero. Assign propositional variables P subsets of S as values, written $V(P)$. Then the truth definition relative to a point x in S is defined inductively as follows:

1. $x \models P \Leftrightarrow x \in V(P)$, where P is a variable;
2. $x \models A \wedge B \Leftrightarrow x \models A$ and $x \models B$;
3. $x \models A \vee B \Leftrightarrow x \models A$ or $x \models B$;
4. $x \models A \rightarrow B \Leftrightarrow \forall y \, [\, y \models A \Rightarrow x \cup y \models B]$;

A formula A of positive relevance logic is valid in this semantics if $0 \models A$ for every assignment to its variables in a semilattice with zero.

This semantics validates all of the formulas of positive **R**. When I first discovered the semantics in 1971, I thought that it was complete for this system. However, Robert K. Meyer and J.M. Dunn soon discovered a formula that is valid for this semantics, but not provable in positive **R**:

$$[(A \to (B \lor C)) \land (B \to D)] \to (A \to (D \lor C))$$

The semilattice system **S** proved somewhat tricky to axiomatize. The problem of axiomatization was solved by Kit Fine, who published his solution as an abstract in 1976. Fine's axiomatization includes a rather complicated rule; a detailed version of his completeness proof was published by Gerald Charlwood in 1981.

The axiomatic version of the semilattice system appears rather complicated and artificial. There are, however, two alternative ways to present the system that make it appear much more natural. The first way is to use an extension of the Fitch-style natural deduction system of Anderson and Belnap; this extension uses a strengthened version of the disjunction rules presented in the first volume of *Entailment*.

The original rule of Anderson and Belnap is as follows:

$$
\begin{array}{l|l}
1 & (P \vee Q)_x \\
\hline
2 & \quad \left| \; P_{\{k\}} \right. \\
\vdots & \quad \left| \; \vdots \right. \\
n-1 & \quad \left| \; R_{y \cup \{k\}} \right. \\
n & \quad \left| \; Q_{\{l\}} \right. \\
\vdots & \quad \left| \; \vdots \right. \\
m & \quad \left| \; R_{y \cup \{l\}} \right. \\
m+1 & R_{x \cup y} \qquad\qquad \vee\mathrm{E},\ 1,\ 2\text{–}(n-1),\ n\text{–}m
\end{array}
$$

This rule has the drawback that the distribution axiom does not follow from the introduction and elimination rules for conjunction and disjunction but has to be "put in by hand."

The new rule of $\vee$ elimination is as follows:

$$
\begin{array}{l|l}
1 & (P \vee Q)_x \\
\hline
2 & \quad \begin{array}{|l} P_x \\ \hline \vdots \\ R_{x \cup y} \end{array} \\
\vdots & \\
n-1 & \\
n & \quad \begin{array}{|l} Q_x \\ \hline \vdots \\ R_{x \cup y} \end{array} \\
\vdots & \\
m & \\
m+1 & R_{x \cup y} \qquad\qquad \vee\text{E}, 1, 2\text{–}(n-1), n\text{–}m
\end{array}
$$

With this new natural deduction rule for disjunction, the distribution axiom follows unproblematically from the introduction and elimination rules.

The fact that this system is a very natural extension of the pure theory of relevant implication appears also from the fact that it coincides with the system of positive relevant implication defined by a set of rules given by Dag Prawitz in Chapter VII of his well known monograph on natural deduction.

Problem

Is the positive semilattice system **S** *decidable?*

Although this problem appears to be open, one can make various remarks about it.

First, the undecidability proofs for **R**, **E** and other classical systems of relevance logics do not seem to adapt to the positive semilattice system. This is because the Meyer-Dunn formula above is not valid in the models constructed from projective spaces that are used in the undecidability proof.

Second, if it is indeed decidable (and I venture the conjecture that it is), then the decision procedure cannot be primitive recursive. This follows from the fact that the implication-conjunction fragment of the system is the same as that of **R**. In a paper of 1999, I showed that there is no primitive recursive decision procedure for this fragment.

The decision problem for $\mathbf{R}_{\rightarrow}$ was solved by Saul Kripke in 1959. It was extended to all of **LR** (**R** without the distribution axiom) by Robert K. Meyer in his doctoral thesis. Kripke's decision procedure is based on a cutfree sequent system for $\mathbf{R}_{\rightarrow}$, together with a lemma that is equivalent to Dickson's Lemma in the theory of polynomial ideals.

If we could extend the cutfree sequent system to **S**, then we might be able to extend the decision procedure as well. Unfortunately, at the moment, the only cutfree system known for **S** uses subscripted formulas and a decision procedure based on this subscripted system is not known.

The subscripted sequent system for **S** has as axioms all sequents of the form

$$\Gamma, A_x \vdash A_x, \Delta.$$

The rules of inference are as follows:

$$\frac{\Gamma \vdash A_y, \Delta \quad B_{x\cup y}, \Theta \vdash \Lambda}{\Gamma, \Theta, A \to B_x \vdash \Delta, \Lambda}\,(\to\vdash) \qquad \frac{\Gamma, A_{\{k\}} \vdash B_{x\cup\{k\}}, \Delta}{\Gamma, \vdash A \to B_x, \Delta}\,(\vdash\to)$$

$$\frac{\Gamma \vdash \Delta, A_x \quad \Theta \vdash B_x, \Lambda}{\Gamma, \Theta \vdash A \wedge B_x, \Delta, \Lambda}\,(\wedge\vdash) \qquad \frac{\Gamma, A_x, B_x \vdash \Delta}{\Gamma, A \wedge B_x \vdash \Delta}\,(\vdash\wedge)$$

$$\frac{\Gamma, A_x \vdash \Delta, \quad \Theta, B_x \vdash \Lambda}{\Gamma, \Theta, A \vee B_x \vdash \Delta, \Lambda}\,(\vee\vdash) \qquad \frac{\Gamma \vdash A_x, B_x, \Delta}{\Gamma, \vdash A \vee B_x, \Delta}\,(\vdash\vee)$$

It is not too hard to show that the theorems of this subscripted sequent system coincide with those of **S**, in the sense that the sequent $\vdash A_{\emptyset}$ is provable in the sequent system if and only if A is provable in **S**.

Problem

Can we base decision procedures on the subscripted sequent systems?

Number of variables for undecidability

I proved in the early 1980s that **R**, **E** and a large family of related relevance logics are undecidable. In my 1984 paper, I sketched a proof that the five variable fragment of **R** is undecidable. Unfortunately, there is a mistake in the proof given there, as I report in a paper published in 2007. The claim, however, is correct (though the proof is in error) and I was able to show in the later paper that the four variable fragment of **R** is undecidable.

Problem

What is the smallest number of variables for which the corresponding fragment of **R** *is undecidable?*

To set the problem in its proper context, let us recall an outline of the undecidability proof for **R**. The construction works by adapting a geometrical definition of multiplication given by von Staudt and employed by von Neumann in his work on continuous geometries.

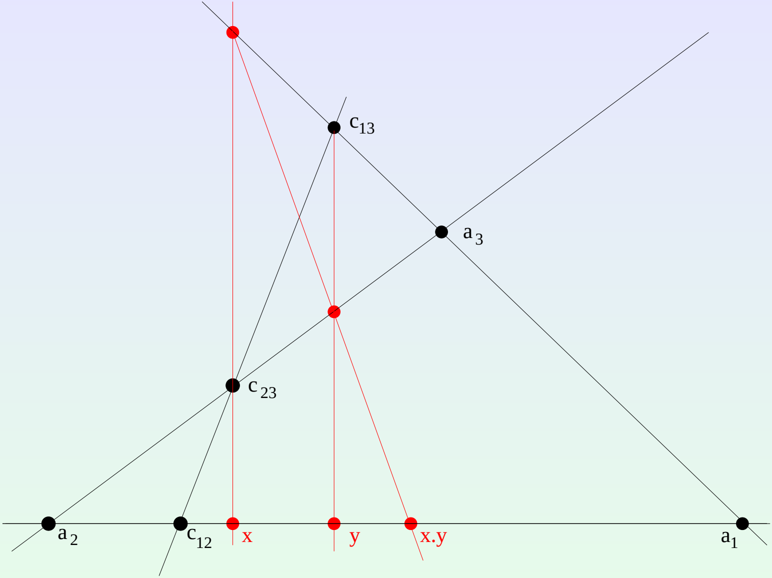


Figure: Multiplication on a line in real projective space

If we assume Desargues's law, then the geometrical multiplication defined in this way is associative.

In a two-dimensional projective space, however, we cannot assume the Desargues law in general, because of the existence of non-Arguesian projective planes. If we add a third dimension to our coordinate frame, however, then we can prove enough of Desargues's law to prove associativity of $x \cdot y$ with appropriate assumptions. This is the construction that proves undecidability for a wide family of relevance logics.

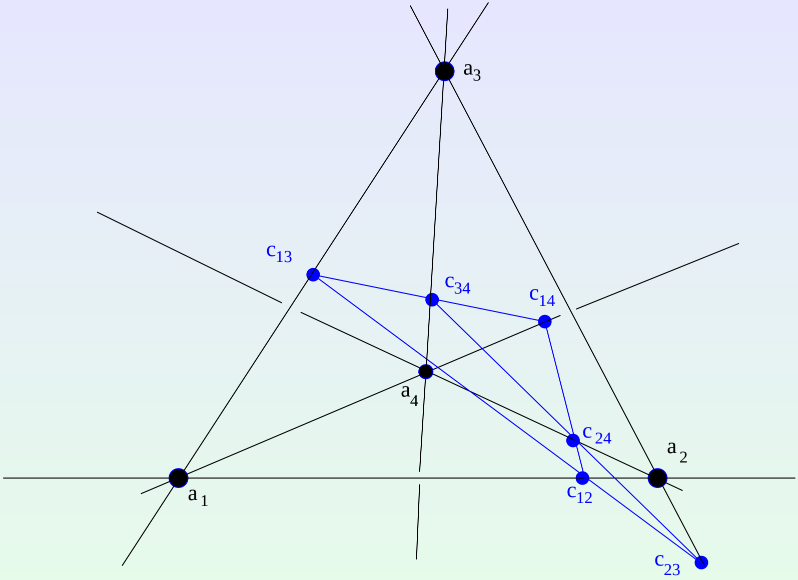


Figure: A 4-frame in real projective space

How do we reduce the number of variables required for undecidability? The trick is to express the elements in a coordinate frame in terms of subsets of the frame. For example, to prove undecidability for the four-variable fragment, we have to find a way to express the elements of a four-frame, together with semigroup elements x, y in terms of four elements definable from the frame. The details can be found in my 2007 paper.

We can add the propositional constants t, f, T, F to **R**, and so it makes sense to talk of the 0-variable fragment of **R**. This fragment is in fact decidable. This follows from the proof by John Slaney that there are exactly 3088 non-equivalent formulas generated from the constants in **R**. This result forms a contrast with linear logic, where the fragment generated by the constants is undecidable, a remarkable result of Max Kanovich.

So, is the answer to our problem one, two, three or four? It is possible that the geometrical technique sketched above can be improved to prove undecidability for three variables. It seems implausible that it could be extended to two. As for the one-variable fragment, it remains deeply mysterious.

Thistlewaite, McRobbie and Meyer in their 1988 monograph suggested a way of proving undecidability for the two-variable fragment of **R**. Let $x \odot y$ be a formula of **R** containing only the two variables x and y. If σ is a term in the language of semigroups, let σ^t be the translation of σ into **R**, using the translation $x \cdot y \mapsto x \odot y$; the translation of a semigroup equality $\sigma = \tau$ is $\sigma^t \leftrightarrow \tau^t$.

Let us say that the formula $\odot$ is a *free associative connective* in **R** if the following holds. If $\Sigma \cup \{\tau\}$ is a finite set of semigroup equalities, then τ is deducible from Σ in the equational theory of semigroups if and only if τ^t is deducible from Σ^t in **R**.

Let us suppose that there is in fact a free associative connective in **R**.

Since there is a finitely presented semigroup in two generators with undecidable word problem, undecidability for formulas with two variables in **R** would follow immediately.

Problem

Is there a free associative connective in **R**?

Somehow, I am inclined to guess that there is no such connective. The encoding of semigroup equations in the undecidability proofs is rather indirect, since the associativity is proved only with respect to a coordinate frame, employing variables additional to those used in encoding the semigroup equations. However if the preceding problem had a positive solution, associativity would have to hold unconditionally.

Complexity of decision problem for $\mathbf{R}_\to$

The decision problem for $\mathbf{R}_\to$ was solved by Saul Kripke in 1959. Kripke's decision procedure appears of high complexity – in fact, the algorithm does not appear to provide an upper bound on the space or time required. The key combinatorial lemma in the proof of correctness is dubbed "Kripke's Lemma" by Anderson and Belnap; it is essentially the same as Dickson's Lemma in number theory and the theory of polynomial ideals. By adapting known bounds for Dickson's Lemma and related problems, it is possible to show Kripke's decision method is primitive recursive in the Ackermann function.

Can we improve on this Ackermann upper bound by getting a tighter estimate of the complexity? I made a small step in this direction by proving that any decision procedure for $\mathbf{R}_{\rightarrow}$ requires exponential space. This was published in a 1990 paper as part of a *festschrift* in honour of my *Doktorvater*, Nuel Belnap. The basic idea is to adapt the exponential space lower bound for the reachability problem for Petri nets (equivalently, vector addition systems) proved by R.J. Lipton.

The lower bound in both cases is proved by encoding bounded counter machines in the system, and depends on the fact that we can define small (linear-size) vector addition systems that generate a doubly exponential number of tokens. This allows the definition of a zero-test for counter machines in which the numbers in the counters are exponentially bounded. Some added complications in the logical case arise from the fact that the unrestricted contraction rule is present.

This still leaves a huge gap between the upper and the lower bounds. I was able to close the gap a few years later, in the case where we include conjunction as well as implication. The same upper bound holds as in the case of pure relevant implication. But in addition, in a paper of 1999 I was able to show that the lower and upper bounds for the system $\mathbf{R}_{\to\wedge}$ essentially coincide, showing that there is no primitive recursive decision procedure for this logic. Thus $\mathbf{R}_{\to\wedge}$ is one of the most complex naturally defined propositional logics.

The lower bound for $\mathbf{R}_{\to\wedge}$ is an adaptation of the undecidability proof of Lincoln, Mitchell, Scedrov and Shankar that linear logic is undecidable. Unfortunately, the proof does not adapt to the pure implicational case, since it depends on the inclusion of additive as well as multiplicative rules.

I made several efforts in the succeeding decades to narrow the gap, but did not succeed. I was planning to present this as an open problem as part of my talk today, but shortly before the Summer of Logic, I was delighted to hear that the problem has been definitively solved by Sylvain Schmitz.

Sylvain Schmitz proves that in fact the pure implicational fragment $\mathbf{R}_{\rightarrow}$ is complete for doubly exponential time, thus solving a problem that has been open for a quarter century. His proof employs branching vector addition systems and builds on earlier results of Demri, Jurdziński, Lachish and Lazić.

Sylvain Schmitz

This brilliant breakthrough result of Schmitz seems an appropriate point to bring my survey to a close. I hope I have inspired some members of the audience to work on these problems! Thank you!


That's all Folks!